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Numerical inverse-scattering-transform analysis of laboratory-generated surface wave trains

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The motion of shallow-water wave trains is studied in the laboratory in one space dimension and one time dimension (1+1). Our focus is on the nonlinear evolution of the wave trains as recorded at five spatially separated probes. Both the linear Fourier transform and the inverse scattering transform (IST) for the periodic Korteweg–de Vries (KdV) equation are exploited in the analysis of the data. IST provides a set of nonlinear basis functions, which are here computed for measured wave data and compared to the sinusoidal basis of the linear Fourier transform. We find that the scattering-transform mode amplitudes are nearly constants of the motion, while the Fourier modes vary substantially in space and time. The results suggest that the KdV equation more nearly describes the nonlinear evolution of the waves than does linear evolution.

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In the analysis of nonlinear wave data one often uses the Fourier transform as a primary analysis tool. One knows of course that the wave motion may well be highly nonlinear and the Fourier method, which is intrinsically linear, may not give results consistent with nonlinear evolution. In the present paper we offer a relatively new approach, based upon a kind of nonlinear Fourier analysis, which arises theoretically as the inverse-scattering-transform (IST) solution of certain nonlinear, integrable wave equations [1–5]. Here we focus on one of these systems, that due to Korteweg and de Vries (KdV), which describes the evolution of long waves in shallow water [6]. IST as applied here is taken to have periodic or quasiperiodic boundary conditions, just as is assumed in the analysis of data using the discrete or fast Fourier transform. Our work is a natural extension of the classic papers by Hammack and Segur [7] who studied KdV evolution in the laboratory for infinite-line boundary conditions and Zabusky and Galvin [8] who studied periodic boundary conditions.

The numerical scattering-transform analysis of periodic or quasiperiodic wave data recorded in the laboratory is the main topic of this paper. The data analysis procedure consists of two steps. First one takes the *direct scattering transform* of the wave train time series in order to compute the associated Floquet spectrum. A separate step, the *inverse scattering transform*, is required to determine the nonlinear, hyperelliptic oscillation modes of the theory, from which the wave train can be reconstructed by a linear superposition law. There are two important features of the approach which make it useful for the analysis of experimental data. The first is that the scattering-transform spectrum is a *higher-order generalization* of the linear Fourier transform. The second feature is that *nonlinear filtering* can be accomplished in the inversion process. These ideas were exploited by Osborne *et al.* [9] to find solitons in complex wave trains in ocean surface wave data.

The dimensional form for the (spacelike) KdV equation is given by

$$\eta_t + c_0\eta_x + \alpha\eta\eta_x + \beta\eta_{xxx} = 0 \ . \tag{1}$$

$\eta(x,t)$ is the wave amplitude which evolves as a function of space x and time t, $c_0=(gh)^{1/2}$, $\alpha=3c_0/2h$, $\beta=c_0h^2/6$; (1) has the linearized dispersion relation $\omega=c_0k-\beta k^3$; g is the acceleration of gravity, c_0 is the linear phase speed, and h is the water depth. Subscripts with respect to x and t refer to partial derivatives. KdV solves a *Cauchy problem:* given an initial wave train, $\eta(x,t=0)$, (1) determines the motion for all time thereafter, $\eta(x,t)$.

Experimentally we record wave amplitudes as a function of time at a single spatial location, thus implying the need to determine the scattering transform of a *time series*, $\eta(0,t)$. For this purpose we employ the *timelike* KdV equation (TKdV) [2]:

$$\eta_x + c_0'\eta_t + \alpha'\eta\eta_t + \beta'\eta_{ttt} = 0 \ , \tag{2}$$

where $c_0'=1/c_0$, $\alpha'=-\alpha/c_0^2$, and $\beta'=-\beta/c_0$; (2) has the linearized dispersion relation $k=\omega/c_0+(\beta/c_0^4)\omega^3$. TKdV solves a boundary value problem: given the temporal evolution $\eta(0,t)$ at some fixed location $x=0$, (2) determines the wave motion over all space $\eta(x,t)$. We assume either periodic $[\eta(x,t)=\eta(x,t+T)]$ or quasiperiodic boundary conditions [there exists a $T(\varepsilon)$ such that $|\eta(x,t+T)-\eta(x,t)|<\varepsilon$ for all t].

Determination of the IST of a periodic, broad-spectrum, discrete wave train $\eta(0,t_n)$, $1\le n\le N$, is now outlined [10–13]. The *IST spectrum* is determined from the one-dimensional Schrödinger eigenvalue problem:

$$\Psi_{tt} + [\lambda\eta(0,t)+\Omega^2]\Psi = 0 \ , \tag{3}$$

where $\lambda=\alpha c_0^2/6\beta$ and Ω is a complex frequency, with

$\Omega^2 = E$ real. Bloch eigenfunction solutions of (3) are periodic or antiperiodic on $0 < t < T$. The trace $\Delta(E)$ of the monodromy matrix $\mathbf{M}$ [which maps solutions of (3) from t to $t+T$] is the Floquet discriminant, $\Delta(E) = \frac{1}{2}\mathrm{Tr}\mathbf{M}$. Solutions of $\Delta(E) = \pm 1$ determine the discrete eigenvalues E_j $(1 \leq j \leq 2N+1)$ which constitute the "main spectrum" of the motion; two adjacent eigenvalues define an "open band" or "degree of freedom" of IST (E_{2j}, E_{2j+1}) when $|\Delta(E)| > 1$; for $E_{2j} = E_{2j+1}$ the band is "degenerate" and the wave amplitude is zero. The *auxiliary spectrum* consists of hyperelliptic functions $\mu_j(x,t)$ which oscillate between the two eigenvalues of an open band according to nonlinear ordinary differential equations discussed elsewhere [10–14]. The width of an open band is the amplitude of a hyperelliptic oscillation mode, $a_j(f_j) = |\mu_j| = (E_{2j+1} - E_{2j})/2\lambda$, e.g., a "single degree of freedom," "spectral component," or "basis state" of KdV; the associated frequencies are given by $f_j = j/T$ (or equivalently $f_j = \Omega_j/\pi$) [15]. A *linear superposition* of these basis states is the solution to the KdV equation:

$$\lambda\eta(x,t) = -E_1 + \sum_{j=1}^{N} [2\mu_j(x,t) - E_{2j} - E_{2j+1}] . \tag{4}$$

In the absence of interactions among nonlinear spectral components, the $\mu_j(x,t)$ degenerate to ordinary cnoidal waves. For small amplitude wave motion the μ_j reduce to sine waves and (4) becomes an ordinary Fourier series [15–17]. Solitons are not present in the data analyzed here and we therefore do not discuss their spectrum.

The determination of the KdV spectrum from (3) [e.g., the modes $a_j(f_j)$] we call the *direct* scattering transform. Determination of the $\mu_j(x,t)$ and the solution to KdV by the linear superposition formula (4) we call the *inverse* scattering transform. Computer algorithms and further mathematical developments are given elsewhere [15–21].

We now discuss the analysis of laboratory-generated surface waves. The data were taken in the wave tank at the Hydraulic Section of the Institute of Civil Engineering, Florence, Italy. The tank has dimensions $0.8 \times 0.76 \times 46$ m^3 and has a programmable wave maker. For the purposes of this paper we conducted a series of experiments, only one of which we report here. The water depth was 49.8 cm (Ursell number $\mathrm{Ur} = 3gH_s T_d^2/4h^2 = 42.7$) and five resistance-gauge probes were placed at a distances of $x_1 = 4.25$, $x_2 = 7.01$, $x_3 = 11.02$, $x_4 = 15.02$, and $x_5 = 19.00$ m from the wave maker. A ramp was placed at the end of the tank from 19 to 50 m, with a slope of 0.02. This configuration allowed waves to be measured in the first 19-m section, with the remaining part of the tank serving as a wave absorber; tests indicate that only about 2% of the energy was reflected from the ramp [22]. Thus the ramp provided an efficient absorber for the long waves under consideration here and consequently minimized reflection. This is a requirement, since the KdV equation and, hence, the scattering transform are unidirectional.

The recorded data are shown in Fig. 1 at each of the five stations; time series of 8 sec duration, recorded 20 times a second, are shown. The wave maker was programmed to produce sinusoidal waves of height 9 cm with a period of 4 sec; Fig. 1 therefore shows two periods of the evolving wave train at each of the five probes. The amplitude-to-depth ratio for the generated waves was 0.09. Clearly the input wave train was distorted significantly from the sinusoidal initial conditions by the time it reached the first probe. Each of the time series was analyzed both by the Fourier and scattering transforms. We found that both methods contained essentially three dominant oscillation modes (other modes tended to be substantially smaller). The scattering transform analysis is shown in Fig. 1(b). The dominant three IST oscillation modes are given at each of the probes, labeled 1–5 in the figure. We have summed the modes and these are shown in Fig. 1(a) (dotted lines). While including more than three modes improves the agreement between the scattering transform and the data, the essential features of the data are well represented by the sum of the three nonlinear oscillation modes. We have effectively filtered out higher-frequency components by summing (2) only over the first three modes.

A comparison with the linear Fourier transform achieves similar good agreement with the data, but the linear modes are found to vary spatially and temporally, while the IST modes are very nearly constant for the

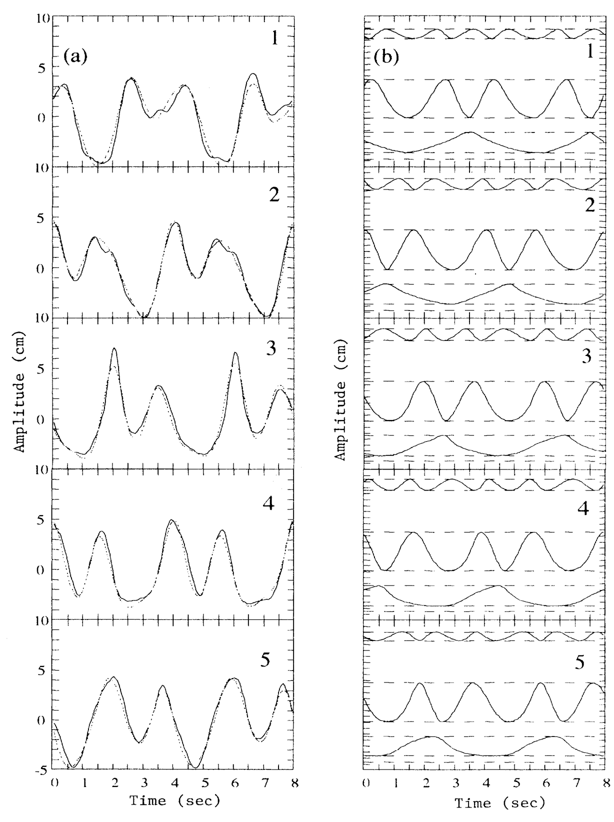

FIG. 1. (a) Wave trains (solid lines) measured at each of five probes at distances 4.25, 7.01, 11.02, 15.02, and 19.00 m from the wave maker. (b) The first three sattering-transform modes at each of the five probes. Sum of the three scattering-transform modes [dotted lines in (a)] gives favorable comparison to the data.

space and time ranges investigated. This is seen in Fig. 2, where we compare the linear Fourier [Fig. 2(a)] and scattering-transform mode amplitudes [Fig. 2(b)] at the five probes. The average of each of the modes was obtained across the five stations and these are represented by the horizontal lines. The uncertainties shown on each mode amplitude are the experimental errors, which we have estimated to be about ± 3.0 mm for each probe; each of the three modes has errors of one-third this amount, ± 1.0 mm. Independent of error estimates, however, the linear Fourier modes are seen to have a strong space-time dynamical behavior because they substantially deviate from their average values as they propagate down the tank. On the other hand, the scattering-transform modes [Fig. 2(b)], within the uncertainties present in the experimental measurements, are practically constant over all the probes. This suggests that the wave evolution is governed primarily by KdV dynamics, e.g., for pure KdV evolution the nonlinear oscillation mode amplitudes are constants of the motion.

It is interesting to observe the shape of the KdV oscillation modes [Fig. 1(b)]. The waves are clearly not sine waves and closer inspection reveals that the modes are also not cnoidal waves, which are the single-mode solution to KdV. Hence, for the case considered here, it is transparent why the natural KdV modes may be viewed as *hyperelliptic* function solutions to the equation, e.g., they are generalizations of the ordinary cnoidal wave. Note further that the KdV modes can be seen to simultaneously translate and to distort from their original shape as they propagate from probe to probe.

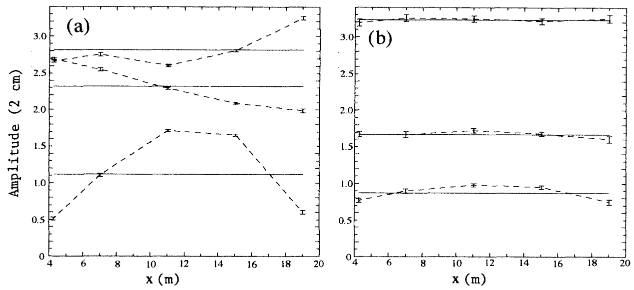

FIG. 2. (a) Amplitudes of first three linear Fourier modes at each of the five probes. (b) Amplitudes of first three scattering-transform modes at each of the five probes. Note that the scattering-transform modes are nearly constant, while the linear Fourier modes substantially vary from probe to probe.

In conclusion, we find that the inverse-scattering-transform spectral decomposition of shallow-water, laboratory-generated unidirectional wave trains discussed herein suggests (1) that KdV evolution dominates the nonlinear dynamics and (2) that the space-time dynamics of the scattering-transform modes is found to be much simpler (nearly constant) than that for the linear Fourier modes.

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